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Effect of Magnetic Field on Nonlinear Interactions of Electromagnetic and Surface Waves in a Plasma Layer

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Investigation is made for nonlinear interaction between incident radiation and a surface wave in a magnetized plasma layer. Both interacting waves are of P polarization. We get the generated currents and fields at combination frequencies analytically. Unlike the S-polarized interacting waves, the magnetic field affects the fundamental waves and leads to an amplification of generated waves when their frequencies approach the cyclotron frequency.

1. INTRODUCTION

The nonlinear generation of waves at harmonics and combination frequencies has been the subject of many theoretical and experimental investigations (see Dolgopolov et al., 1976; Hussein, 1980; El-Siragy et al., 1981; Grozev et al., 1981). This subject of prime importance, especially in plasma diagnostic techniques and in studying the interaction of intensive electromagnetic waves with matter (laser-fusion reaction).

Barakat et al (1975) studied wave generation at combination frequencies for the oblique incidence of P-polarized electromagnetic waves on a semi-bounded, isotropic plasma. The amplitudes and phases of the waves radiated in this case depend neither on the width of the transition layer nor on the plasma density distribution through the transition layer but only on the plasma density in the homogeneous region of the medium. If the two interacting waves are surface waves (Dolgopolov et al., 1976) the change

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in amplitudes of the surface waves at the expense of wave radiation at combination frequencies, under conditions where a weakly nonlinear approximation applies (Barakat et al., 1975) is much slower.

It is of interest to study the problem treated by Dolgopolov et al. (1976) and Barakat et al. (1975) in a new way. We assume that the P-surface wave with frequency ω_1 and P-incident radiation with frequency ω_2 propagate to meet the surface wave along the y axis. It is more reasonable to consider a plasma layer of width a in which the plasma density is an arbitrary differentiable function of x in the region $0 \leq x \leq a$ and is equal to zero elsewhere. Also, an external static magnetic field is applied to the plasma and directed along z axis, $\mathbf{H}_0 = \mathbf{e}_z H_0$. We shall neglect the thermal velocities, the positive ion current, and the pressure gradients.

2. LINEAR APPROXIMATION

In the linear approximation, we have to solve the following set of equations:

$$\begin{aligned} \operatorname{curl} \mathbf{E}_{1,2} &= i\frac{\omega_{1,2}}{c}\mathbf{H}_{1,2} \\ \operatorname{curl} \mathbf{H}_{1,2} &= -i\frac{\omega_{1,2}}{c}\mathbf{E}_{1,2} + \frac{4\pi}{c}\mathbf{J}_{1,2} \\ \mathbf{J}_{1,2} &= -en_0\mathbf{V}_{1,2} \\ \frac{\partial \mathbf{V}_{1,2}}{\partial t} &= -\frac{e}{m}\left(\mathbf{E}_{1,2} + \frac{1}{c}\cdot \mathbf{V}_{1,2} \times \mathbf{H}_0\right) \end{aligned} \tag{1}$$

where we denoted the surface wave by 1 and the incident radiation by 2. The electromagnetic fields of these waves have the form (see Stepanov, 1965)

$$\{\mathbf{E}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t)\} = \{E_1(x), H_1(x)\}e^{i(k_1 y - \omega_1 t)} + \{E_2(x), H_2(x)\}e^{i(k_2 y - \omega_2 t)} + c.c. \tag{2}$$

where

$$H_{(1,2)x} = H_{(1,2)y} = E_{(1,2)z} = 0$$

From these equations, we obtain the following expression for the electron current density components:

$$\begin{aligned} J_{(1,2)x} &= \frac{i}{4\pi}\frac{\omega_{1,2}\omega_p^2}{\omega_{1,2}^2\omega_c^2}\left[E_{(1,2)x} + \frac{i\omega_c}{\omega_{1,2}}E_{(1,2)y}\right] \\ J_{(1,2)y} &= \frac{i}{4\pi}\frac{\omega_{1,2}\omega_p^2}{\omega_{1,2}^2 - \omega_c^2}\left[-\frac{i\omega_c}{\omega_{1,2}}E_{(1,2)x} + E_{(1,2)y}\right] \\ J_{(1,2)z} &= 0 \end{aligned} \tag{3}$$

where $\omega_p \equiv (4\pi e^2 n_0/m)^{1/2}$ is the plasma Langmuir frequency and $\omega_c = (-eH_0/mc)$ is the electron-cyclotron frequency.

For fundamental waves, we can write the electric field components as

$$E_{\alpha x} = -\frac{N_\alpha}{k_\alpha}\frac{1}{\mathscr{E}_{\perp\alpha}}\left(k_\alpha H_{\alpha z} + \frac{\mathscr{E}_{2\alpha}}{\mathscr{E}_{1\alpha}}\frac{\partial H_{\alpha z}}{\partial x}\right), \qquad \alpha \equiv (1,2)$$
$$E_{\alpha y} = -i\frac{N_\alpha}{k_\alpha}\frac{1}{\mathscr{E}_{\perp\alpha}}\left(k_\alpha \frac{\mathscr{E}_{2\alpha}}{\mathscr{E}_{1\alpha}} H_{\alpha z} + \frac{\partial H_{\alpha z}}{\partial x}\right), \qquad N_\alpha \equiv \frac{K_\alpha c}{\omega_\alpha} \tag{4}$$

with

$$\mathscr{E}_{1\alpha} = 1 - \frac{\omega_p^2(x)}{\omega_\alpha^2 - \omega_c^2}, \qquad \mathscr{E}_{\alpha 2} = \frac{\omega_c \omega_p^2(x)}{\omega_\alpha(\omega_\alpha^2 - \omega_c^2)}, \qquad \mathscr{E}_{\perp\alpha} = \frac{\mathscr{E}_{1\alpha}^2 - \mathscr{E}_{2\alpha}^2}{\mathscr{E}_{1\alpha}} \tag{5}$$

For the magnetic field component of the fundamental P-polarized surface and incident waves, we can easily get the following differential equation:

$$\mathscr{E}_{\perp\alpha}\frac{\partial}{\partial x}\left(\frac{1}{\mathscr{E}_{\perp\alpha}}\frac{\partial H_{\alpha z}}{\partial x}\right) + \varkappa_\alpha^2 H_{\alpha z} = 0 \tag{6}$$

where

$$\varkappa_\alpha^2 \equiv \frac{\omega_\alpha^2}{c^2}\mathscr{E}_{\perp\alpha} - k_\alpha^2 - k_\alpha \mathscr{E}_{\perp\alpha}\frac{\partial}{\partial x}\frac{\mathscr{E}_{2\alpha}}{\mathscr{E}_{1\alpha}\mathscr{E}_{\perp\alpha}}$$

and for surface wave $\varkappa_1^2 < 0$, while for incident radiation $\varkappa_2^2 > 0$.

The solutions of (6) are as follows: (i) for surface wave (ω_1, k_1):

$$H_{1z} = H_{01}e^{\varkappa_{01}x}, \quad (x \le 0); \qquad H_{1z} = H_{a1}e^{-\varkappa_{01}(x-a)} \quad (x \ge a)$$
$$H_{1z} = H_{01}\left\{1 + \int_0^x \mathscr{E}_{\perp 1}\,dx' \int_0^{x'} \frac{\varkappa_1^2}{\mathscr{E}_{\perp 1}}\,dx''\right\} + H_{01}\varkappa_{01}\int_0^x \mathscr{E}_{\perp 1}\,dx', \qquad (0 \le x \le a) \tag{7}$$

$\varkappa_{0\alpha}$ is the decay coefficient in vacuum, $\varkappa_{0\alpha}^2 = \omega_\alpha^2/c^2 - k_\alpha^2$.

(ii) For incident radiation (ω_2, k_2):

$$H_{2z} = H_{02}(e^{i\varkappa_{02}x} - re^{-i\varkappa_{02}x}) \qquad \text{at } x \le 0$$
$$H_{2z} = H_{\alpha 2}e^{i\varkappa_{02}(x-a)} \qquad \text{at } x \ge a$$
$$H_{2z} = H_{02}\left\{(1-r)\left[1 - \int_0^x \mathscr{E}_{\perp\alpha}\,dx' \int_0^{x'} \frac{\varkappa_2^2}{\mathscr{E}_{\perp 2}}\,dx''\right] + i\varkappa_{02}(1+r) \times\left[\int_0^x \mathscr{E}_{\perp 2}\,dx' - \int_0^x \mathscr{E}_{\perp 2}\,dx' \int_0^{x'} \frac{\varkappa_2^2}{\mathscr{E}_{\perp 2}}\,dx'' \int_0^{x''} \mathscr{E}_{\perp 2}^2\,dx''\right]\right\} \quad \text{at } 0 \le x \le a \tag{8}$$

where r is the reflection coefficient of the plasma layer, given by

$$r=\frac{\int_0^a (\varkappa_2^2/\mathscr{E}_{\perp 2})\,dx-\varkappa_{02}^2\int_0^a \mathscr{E}_{\perp 2}\,dx}{2i\varkappa_{02}+\varkappa_{02}^2\int_0^a \mathscr{E}_{\perp 2}\,dx+\int_0^a (\varkappa_2^2/\mathscr{E}_{\perp 2})\,dx} \tag{9}$$

3. GENERATION WITH FREQUENCIES

Let us now look for the generated waves at combination frequencies due to the nonlinear interaction of both P-polarized incident radiation with surface waves.

The terms $n_1\mathbf{V}$, $(\mathbf{V}\cdot\nabla)\mathbf{V}$ and $\mathbf{V}\times\mathbf{H}$ are quadratic with respect to the physical variables; therefore they are the nonlinear terms we take into account now.

In the next approximation of the perturbation theory, the velocity components of generated waves with combination frequencies $\omega'=\omega_1+\omega_2$ are

$$\begin{aligned} V'_x&=(\omega'^2-\omega_c^2)^{-1}(i\omega'\Delta_a-\omega_c\Delta_b)\\ V'_y&=(\omega'^2-\omega_c^2)^{-1}(-\omega_c\Delta_a+i\omega'\Delta_b)\\ V'_z&=0 \end{aligned} \tag{10}$$

where

$$\begin{aligned} \Delta_a&\equiv-\frac{e}{m}E'_x-\frac{e^2}{mc}\sum_{\substack{\alpha_1\beta=1,2\\ \alpha\neq\beta}}\frac{\omega_\alpha}{m(\omega_\alpha^2-\omega_c^2)}\left(\frac{\omega_c}{\omega_\alpha}E_{\alpha x}-iE_{\alpha y}\right)H_{\beta z}\\ \Delta_b&\equiv-\frac{e}{m}E'_y-\frac{e^2}{mc}\sum_{\substack{\alpha_1\beta=1,2\\ \alpha\neq\beta}}\frac{\omega_\alpha}{m(\omega_\alpha^2-\omega_c^2)}\left(iE_{\alpha x}-\frac{\omega_c}{\omega_\alpha}E_{\alpha y}\right)H_{\beta z} \end{aligned} \tag{11}$$

and it is clear that V'_z is independent on the field components of the fundamental waves; therefore we assume that the z component of the electric field of generated waves is to be zero.

From the conservation law of density; the perturbed electron density takes the form

$$n_\alpha=-\frac{e}{m(\omega_\alpha^2-\omega_c^2)}\left\{ik_\alpha n_0\left(E_{\alpha y}-\frac{i\omega_c}{\omega_\alpha}E_{\alpha x}\right)+\frac{\partial}{\partial x}\left[n_0\left(E_{\alpha x}+\frac{i\omega_c}{\omega_\alpha}E_{\alpha y}\right)\right]\right\} \tag{12}$$

The components of the current density of the generated waves can be obtained in the form

$$
\begin{aligned}
J'_x &= \frac{\omega_p^2}{4\pi(\omega'^2-\omega_c^2)}\,(i\omega' E'_x - \omega_c E'_y) + J''_x \\
J'_y &= \frac{\omega_p^2}{4\pi(\omega'^2-\omega_c^2)}\,(i\omega' E'_y + \omega_c E'_x) + J''_y
\end{aligned} \tag{13}
$$

where the nonlinear additions J''_x and J''_y are given by

$$
J''_x = \frac{\omega_p^2}{4\pi(\omega'^2-\omega_c^2)}\left(\frac{e}{mc}\right)\sum_{\substack{\alpha,\beta'\\ \alpha\neq\beta}}^{1,2}\frac{\omega_\alpha}{(\omega_\alpha^2-\omega_c^2)}\left[\frac{i\omega_c\omega_\beta}{\omega_\alpha}E_{\alpha x} + \left(\omega' + \frac{\omega_c^2}{\omega_\alpha}\right)E_{\alpha y}\right]H_{\beta z}
$$

$$
+\frac{1}{4\pi}\left(\frac{\omega_p^2}{n_0}\right)\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}^{1,2}\frac{\omega_\alpha}{\omega_\beta(\omega_\alpha^2-\omega_c^2)}\left(E_{\alpha x} + \frac{i\omega_c}{\omega_\alpha}E_{\alpha y}\right)\operatorname{div} n_0\mathbf{V}_\beta
$$

$$
J''_y = \frac{\omega_p^2}{4\pi(\omega'^2-\omega_c^2)}\left(\frac{e}{mc}\right)\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}^{1,2}\frac{\omega_\alpha}{(\omega_\alpha^2-\omega_c^2)}
$$

$$
\times\left[\left(\frac{\omega_c^2}{\omega_\alpha} - \omega'\right)E_{\alpha x} - i\omega_c\left(2 + \frac{\omega_\beta}{\omega_\alpha}\right)E_{\alpha y}\right]H_{\beta z}
$$

$$
+\frac{1}{4\pi}\left(\frac{\omega_p^2}{n_0}\right)\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}^{1,2}\frac{\omega_\alpha}{\omega_\beta(\omega_\alpha^2-\omega_c^2)}\left(E_{\alpha y} - \frac{i\omega_c}{\omega_\alpha}E_{\alpha x}\right)\operatorname{div} n_0\mathbf{V}_\beta
$$

From Maxwell's equations (13) we derive the following expressions for the generated electric wave components and the equation of the magnetic field of the P-polarized wave at combination frequencies:

$$
\begin{aligned}
E'_x &= -i\frac{c}{\omega'\mathscr{E}'_\perp}\left(k'H'_z + \frac{\mathscr{E}'_2}{\mathscr{E}'_1}\frac{\partial H'_z}{\partial x} + \frac{4\pi i}{c}J''_x + \frac{4\pi}{c}\frac{\mathscr{E}'_2}{\mathscr{E}'_1}J''_y\right) \\
E'_y &= -i\frac{c}{\omega'\mathscr{E}'_\perp}\left(k'\frac{\mathscr{E}'_2}{\mathscr{E}'_1}H'_z + \frac{\partial H'_z}{\partial x} + \frac{4\pi i}{c}\frac{\mathscr{E}'_2}{\mathscr{E}'_1}J''_x + \frac{4\pi}{c}J''_y\right)
\end{aligned} \tag{14}
$$

$$
\mathscr{E}'_\perp\frac{\partial}{\partial x}\left(\frac{1}{\mathscr{E}'_\perp}\frac{\partial H'_z}{\partial x}\right) + \varkappa'^2 H'_z = R'(x) \tag{15}
$$

where

$$
\mathscr{E}'_1 \equiv 1 - \frac{\omega_p^2}{\omega'^2-\omega_c^2}, \qquad \mathscr{E}'_2 \equiv \frac{\omega_c\omega_p^2}{\omega'(\omega'^2-\omega_c^2)}, \qquad \mathscr{E}'_\perp \equiv \frac{\mathscr{E}'^2_1 - \mathscr{E}'^2_2}{\mathscr{E}'_1}
$$

$$
\varkappa'^2 \equiv \frac{\omega'^2}{c^2}\mathscr{E}'_\perp - k'^2 - k'\mathscr{E}'_\perp\frac{\partial}{\partial x}\frac{\mathscr{E}'_2}{\mathscr{E}'_1\mathscr{E}'_\perp}, \qquad k' = k_1 + k_2
$$

$$R'(x) \equiv -\frac{4\pi}{c}\mathscr{E}'_{\perp}\left[\text{curl}\,\frac{J''}{\mathscr{E}'_{\perp}}\right]_z - \frac{4\pi i}{c}\mathscr{E}'_{\perp}\left[\text{div}\,\frac{\mathscr{E}'_2}{\mathscr{E}'_1\mathscr{E}'_{\perp}}\right]$$

From (14) and (15), the generated waves are of P polarization, and it can be either surface waves or radiation oscillates in both directions from the plasma layer. This will depend on the sign of $\varkappa'^2$; i.e., if

$$k'^2 < \frac{\omega'^2}{c^2}\mathscr{E}'_{\perp} - k'\mathscr{E}'_{\perp}\frac{\partial}{\partial x}\frac{\mathscr{E}'_2}{\mathscr{E}'_1\mathscr{E}'_{\perp}}$$

we have electromagnetic radiation; if

$$k'^2 > \frac{\omega'^2}{c^2}\mathscr{E}'_{\perp} - k'\mathscr{E}'_{\perp}\frac{\partial}{\partial x}\frac{\mathscr{E}'_2}{\mathscr{E}'_1\mathscr{E}'_{\perp}}$$

we have surface wave.

Let us consider the case of electromagnetic radiation; therefore in vacuum regions [$\omega_p = 0$, $\mathscr{E}'_1 = \mathscr{E}'_{\perp} = 1$, $\mathscr{E}'_{2=0}$ and hence $R'(x) = 0$]. The solutions of (5) are

$$\begin{aligned} H'_z &= H'_0\, e^{-i\varkappa'_0 x} && \text{for } x \le 0 \\ H'_z &= H'_a\, e^{i\varkappa'_0(x-a)} && \text{for } x \ge a \end{aligned} \tag{16}$$

where $\varkappa_0'^2 \equiv (\omega'^2/c^2) - k'^2$ and H'_0, H'_a are the ampoitudes of the generated waves in vacuum.

From Ginzburg (1960), we shall use the method of successive approximation to obtain the solution of (15) in the plasma layer ($0 \le x \le a$). Let us set $H'_z = H_z'^{(0)} + H_z'^{(1)} + \cdots$, and consider the characteristic lengths of the waves under investigation are large compared to the width of the plasma layer, i.e., $|\varkappa' a| \ll 1$. Therefore we get

$$\begin{aligned} H'_z = H'_0&\left[1 - \int_0^x \mathscr{E}'_{\perp}\,dx' \int_0^{x'} \frac{\varkappa'^2}{\mathscr{E}'_{\perp}}\,dx''\right] \\ &- i\varkappa'_0 H'_0\left[\int_0^x \mathscr{E}'_{\perp}\,dx' - \int_0^x \mathscr{E}'_{\perp}\,dx' \int_0^{x'} \frac{\varkappa'^2}{\mathscr{E}'_{\perp}}\,dx'' \int_0^{x''} \mathscr{E}'_{\perp}\,dx'''\right] \\ &+ \int_0^x \mathscr{E}'_{\perp}\,dx' \int_0^{x'} \frac{R'}{\mathscr{E}'_{\perp}}\,dx'' \end{aligned} \tag{17}$$

From the conditions of continuity of the functions H'_z and $\partial H'_z/\partial x$ at points $x = 0$ and $x = a$, we find the amplitudes of the generated waves at combination frequencies as

$$H'_0 \equiv \frac{\int_0^a \frac{R'}{\mathscr{E}'_\perp}\,dx - i\varkappa'_0 \int_0^a \mathscr{E}'_\perp\,dx \int_0^x \frac{R'}{\mathscr{E}'_\perp}\,dx'}{2i\varkappa'_0 + \varkappa'^2_0 \int_0^a \mathscr{E}'_\perp\,dx + \int_0^a \frac{\varkappa'^2}{\mathscr{E}'_\perp}\,dx} \tag{18}$$

$$H'_a \equiv \left(1 - i\varkappa'_0 \int_0^a \mathscr{E}'_\perp\,dx\right) H'_0 + \int_0^a \mathscr{E}'_\perp\,dx \int_0^x (R'\mathscr{E}'_\perp)\,dx' \tag{19}$$

4. DISCUSSIONS AND CONCLUSIONS

To get a clearer picture and the physical meaning of the results, it is necessary to evaluate the integrals $\int_0^a (R'/\mathscr{E}'_\perp)\,dx$ and $\int_0^a \mathscr{E}'_\perp\,dx \int_0^x (R'/\mathscr{E}'_\perp)\,dx'$. These are in fact too long and complicated and depend on the density profile, which in our case is assumed to be arbitrary.

For low-density plasma ($\omega p \ll \omega_2$), the reflection coefficient r can be neglected compared to unity. Besides, the quantities $\mathscr{E}'_1$, $\mathscr{E}'_\perp$, $\mathscr{E}_1$, and $\mathscr{E}_\perp$ become close to unity. In this case, the generated amplitudes are approximately the same and radiated from the plasma layer in both directions, and they are given by

$$H'_0 \simeq H'_a \simeq \left(\frac{2\pi}{c}\right)\left(\frac{K'}{\varkappa'_0}\right) \int_0^a \left(J''_x - \mathscr{E}'_2 J''_y\right) dx \tag{20}$$

It is clear that the amplitude of the generated waves in rarefied plasma is generally proportional to the plasma density in the layer considered, also on the ratio (K_2/K_1).

For normal incidence of radiation, $K_2 = 0$ and it is clear from (20) that $K' = K_1$ and the amplitudes are reduced.

Unlike the previous work mentioned in the Introduction, we find that an applied external magnetic field may lead to a sharp increase in the generated amplitudes [equations (18), (19)], especially near resonance, i.e., at $\omega_1 + \omega_2 \sim \omega_c$, $\omega_1 \sim \omega_c$, $\omega_2 \sim \omega_c$.

In case of isotropic plasma $\omega_c = 0$, and if we consider a semibounded plasma instead of a layer (i.e., if the plasma density is constant in the region $x \geq a$), one can easily obtain expressions for H'_0 and H'_a which agree with relations (15) of Dolgopolov *et al.* (1976).

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